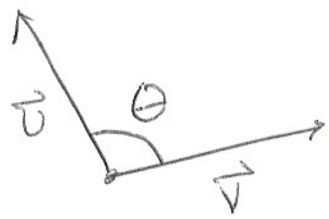
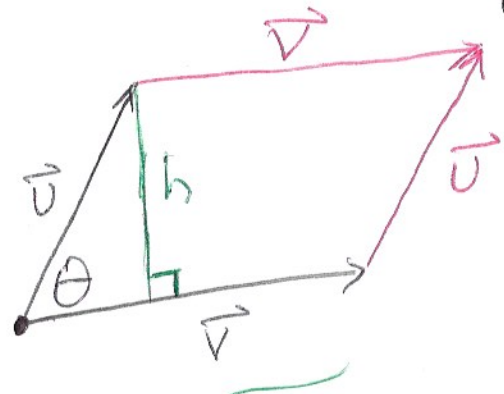
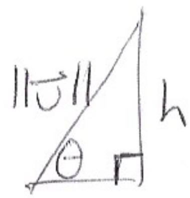
$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$ = AREA OF PARALLELOGRAM WITH $\vec{u}, \vec{v}$ AS ADJACENT SIDES







$b = \|\vec{v}\|$

$\sin\theta = \dfrac{h}{\|\vec{u}\|}$

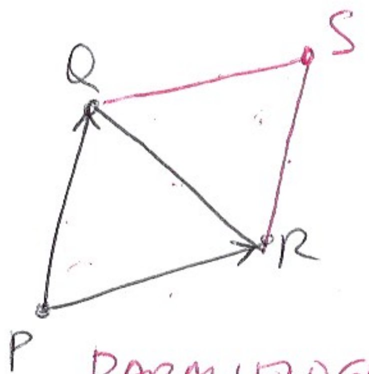
$A = bh$

$h = \|\vec{u}\|\sin\theta$

$A = \|\vec{v}\|\|\vec{u}\|\sin\theta = \|\vec{u} \times \vec{v}\|$

eg. FIND THE AREA OF A TRIANGLE WITH VERTICES

$P(-2,-1,0)$ $Q(1,1,3)$ $R(0,-2,1)$



$\frac{1}{2}$ AREA OF ▱ PQSR = AREA OF $\triangle PQR$

$\overrightarrow{PQ} = \langle 3,2,3 \rangle$ $\overrightarrow{PR} = \langle 2,-1,1 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ 3 & 2 \\ 2 & -1 \end{matrix} = \begin{matrix} 2\vec{i} + 6\vec{j} - 3\vec{k} \\ +3\vec{i} - 3\vec{j} - 4\vec{k} \end{matrix}$$

$= \langle 5,3,-7 \rangle$

$\langle 5,3,-7 \rangle \cdot \langle 3,2,3 \rangle = 15 + 6 - 21 = 0$

$\langle 5,3,-7 \rangle \cdot \langle 2,-1,1 \rangle = 10 - 3 - 7 = 0$

MANDATORY SANITY CHECK:

CHECK $(\vec{u} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{v}) \cdot \vec{v} = 0$

$$\text{AREA } \triangle PQR = \frac{1}{2}\|\langle 5, 3, -7\rangle\|$$
$$= \frac{1}{2}\sqrt{5^2 + 3^2 + (-7)^2}$$
$$= \frac{1}{2}\sqrt{25+9+49}$$
$$= \frac{1}{2}\sqrt{83} = \frac{\sqrt{83}}{2}$$

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$
$$k(\vec{u} \times \vec{v}) = k\vec{u} \times \vec{v} = \vec{u} \times k\vec{v}$$
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$
$$(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$$
$$\vec{0} \times \vec{u} = \vec{u} \times \vec{0} = \vec{0} \longrightarrow \|\vec{0} \times \vec{u}\| = \overset{0}{\cancel{\|\vec{0}\|}}\|\vec{u}\|\sin\theta = 0$$
$$\vec{u} \times \vec{u} = \vec{0} \longrightarrow \|\vec{u} \times \vec{u}\| = \|\vec{u}\|\|\vec{u}\|\overset{0}{\cancel{\sin\theta}} = 0$$

eg. $\langle 12, -18, 15\rangle \times \langle -9, \frac{27}{2}, -\frac{45}{4}\rangle$

$$= 3\langle 4, -6, 5\rangle \times (-3)\langle 3, -\frac{9}{2}, \frac{15}{4}\rangle$$
$$= 3\langle 4, -6, 5\rangle \times (-3)(\tfrac{1}{4})\langle 12, -18, 15\rangle$$
$$= \langle 12, -18, 15\rangle \times (-\tfrac{3}{4})\langle 12, -18, 15\rangle$$
$$= -\frac{3}{4}(\langle 12, -18, 15\rangle \times \langle 12, -18, 15\rangle$$
$$= -\frac{3}{4} \cdot \vec{0}$$
$$= \vec{0}$$

IS $(\vec{U}\times\vec{V})\times\vec{W} = \vec{U}\times(\vec{V}\times\vec{W})$? NO

VECTOR X VECTOR VECTOR X VECTOR

LEGAL-VECTOR LEGAL-VECTOR

VECTOR ADDITION/SUBTRACTION
SCALAR MULTIPLICATION
DOT PRODUCT
CROSS PRODUCT

IS $(\vec{U}\cdot\vec{V})\times\vec{W}$ VALID ? NO

SCALAR X VECTOR

ILLEGAL

IS $(\vec{U}\times\vec{V})\cdot(\vec{W}\times\vec{t})$ VALID ? YES, PRODUCES A SCALAR

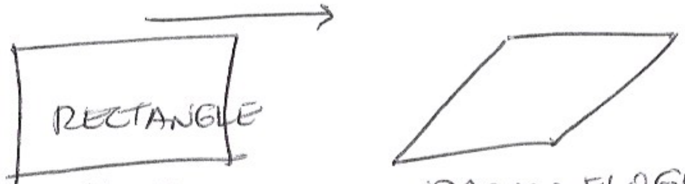
VECTOR · VECTOR

LEGAL - SCALAR

$(\vec{U} \times \vec{V}) \cdot \vec{W} = \vec{U} \cdot (\vec{V} \times \vec{W})$ IS CALLED THE TRIPLE SCALAR PRODUCT OF $\vec{U}, \vec{V}, \vec{W}$

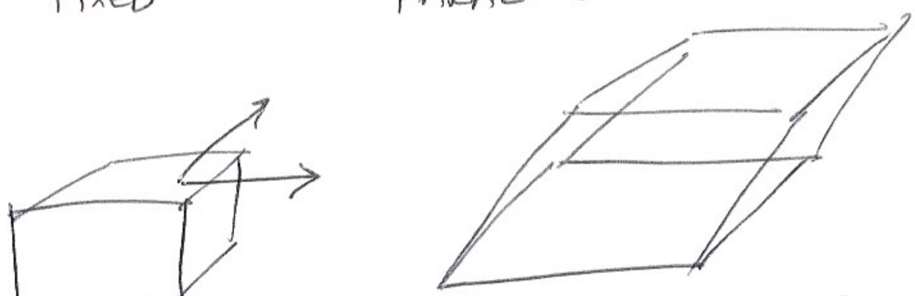
VECTOR · VECTOR
LEGAL-SCALAR

VECTOR · VECTOR
LEGAL-SCALAR

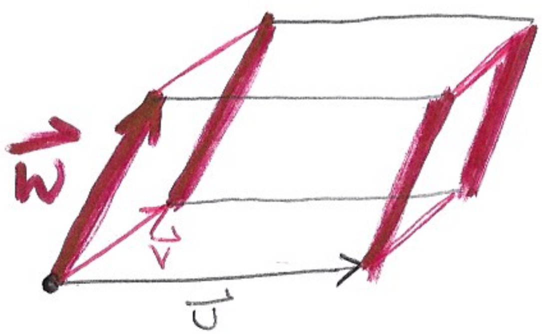


FIXED

PARALLELOGRAM



PARALLELEPIPED



VOLUME OF PARALLELEPIPED

$= \left| \vec{U} \cdot (\vec{V} \times \vec{W}) \right|$

OR $\left| (\vec{U} \times \vec{V}) \cdot \vec{W} \right|$

FIND THE VOLUME OF PARALLELEPIPED WITH ADJACENT EDGES $\langle 3,1,-2\rangle = \vec{u}$

$\langle 1,0,-1\rangle = \vec{v}$

$\langle -1,-1,2\rangle = \vec{w}$

$$\vec{u}\times\vec{v} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 3 & 1 & -2 & 3 & 1 \\ 1 & 0 & -1 & 1 & 0 \end{array}\right| = -\vec{i} - 2\vec{j} + 3\vec{j} - \vec{k} = \langle -1, 1, -1\rangle$$

MANDATORY SANITY CHECK

$\langle -1,1,-1\rangle \cdot \langle 3,1,-2\rangle = -3+1+2=0$ ✓

$\langle -1,1,-1\rangle \cdot \langle 1,0,-1\rangle = -1+0+1=0$

$$\left|(\vec{u}\times\vec{v})\cdot\vec{w}\right| = \left|\langle -1,1,-1\rangle\cdot\langle -1,-1,2\rangle\right|$$

$$= |1-1-2|$$

$$= |-2| = 2$$

SHORTCUT

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \left|\begin{array}{ccc|cc} 3 & 1 & -2 & 3 & 1 \\ 1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 2 & -1 & -1 \end{array}\right| = \begin{array}{l} 0 + \cancel{1} + 2 \\ -2 - \cancel{3} - 0 \end{array} = -2$$